我对线性代数课程的点滴认识

林开亮

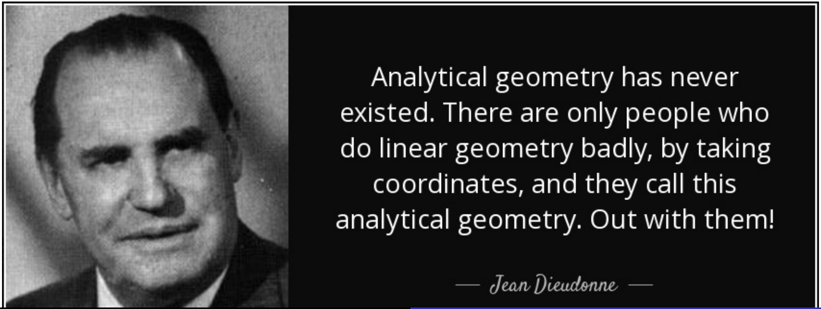
西北农林科技大学理学院

2023 年 12 月 17 日

Dieudonné(1906–1992)

There is hardly any theory which is more elementary [than linear algebra], in spite of the fact that generations of professors and textbook writers have obscured its simplicity by preposterous calculations with matrices.

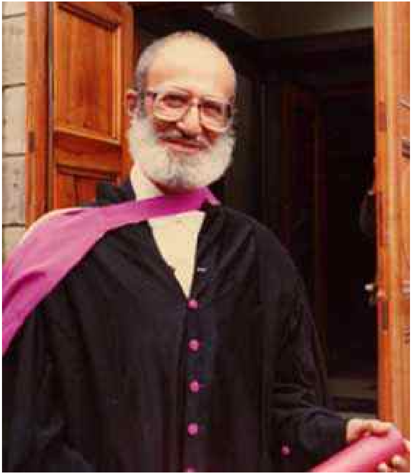
Dieudonné, Foundations of Modern Analysis, Vol. 1



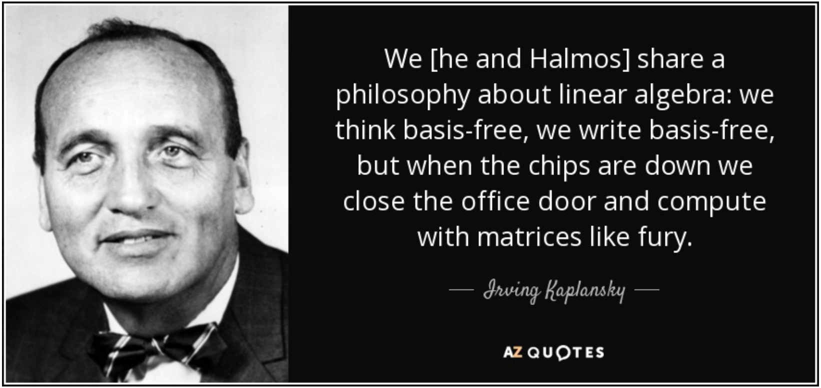
It turned out, however, that there is a part of mathematics we both know and like. It is a small subject, not considered deep. It has, they say, no intrinsic importance; it is merely a useful tool and an occasional source of examples in other subjects. Both Jack and I tend to be shamefaced about admitting we like it; it is a little like admitting that you read westerns. The subject I am talking about is linear algebra. Jack and I enjoy finding and sharing puzzles about matrices. Once we found that out, we found other common interests and tastes, in food, in walking, in music, and in literature, and we enjoy each other's company. Jack has forgiven me the initial hullabaloo of my Michigan years; anybody who understands and likes the Jordan canonical form can't be all bad.

Halmos, I Want to be a Mathematician

Halmos(1916–2006)



Kaplansky(1917–2006)



Linear algebra did not really come to be recognized as a subject until the 1930s. Particularly influential in this process were the book of B. L. van der Waerden from 1930 to 1931 and the book of Garrett Birkhoff and Saunders MacLane of 1941. Both were on "Modern Algebra", but included chapters on linear algebra. Historian Jean-Luc Dorier regards Paul Halmos' book Finite Dimensional Vector Spaces, first published in 1942, as the first book about linear algebra written for undergraduates.

Carl C. Cowen, On the Centrality of Linear Algebra in the Curriculum

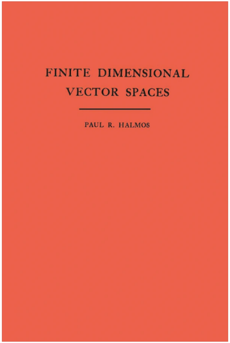
《有限维向量空间》

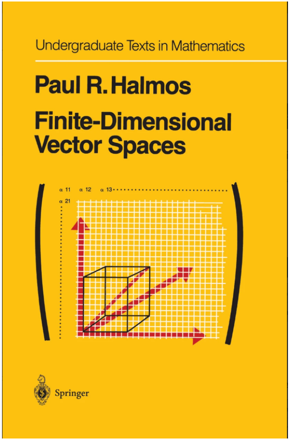
Mathematics

Finite Dimensional Vector Spaces. (AM-7), Volume 7

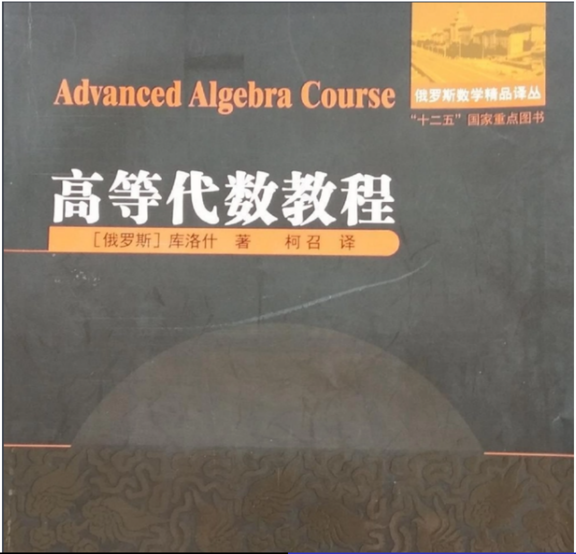
Paul R. Halmos

Series:
Annals of Mathematics Studies



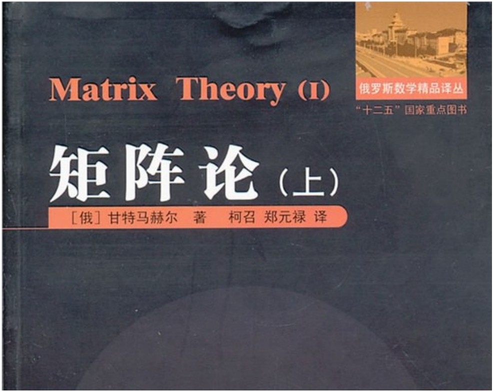


Kurosh(1908–1971)《高等代数教程》(1946)



Malcev(1909–1967) 的《线性代数基础》(1948)

Gantmacher(1908–1964) 的《矩阵论》(1953)

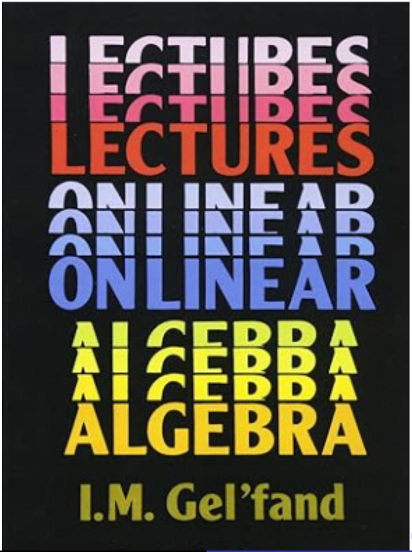


柯召 (1910–2002)

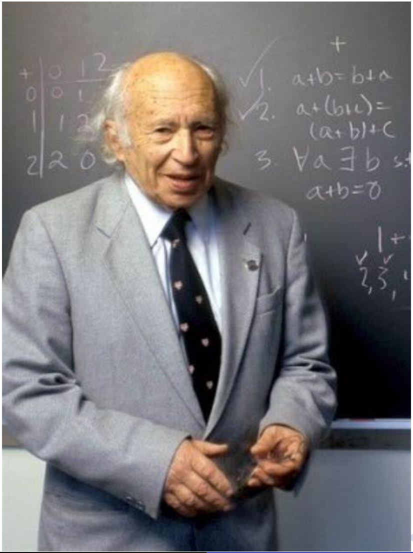

书海中的柯召
1 2 3
大学的设备，不如师资重要，西南联合大学就是证明。它的设备不行，还是培养杨振宁、李政道等多位杰出学者，其原因是西南联合大学的师资力量很强。
—— 柯召

郑元禄 (1938–)

Gelfand 的《线性代数讲义》(1948)

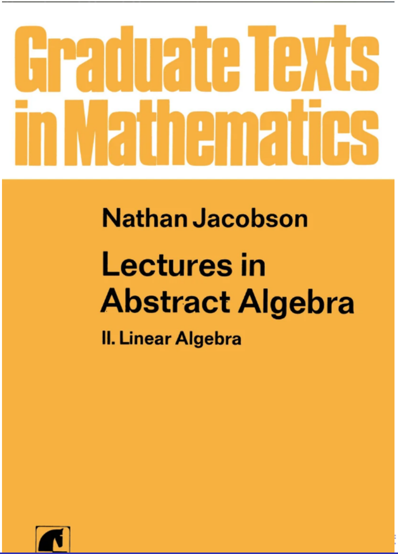


Gelfand(1913–2009)



刘亦珩 (1904–1967)

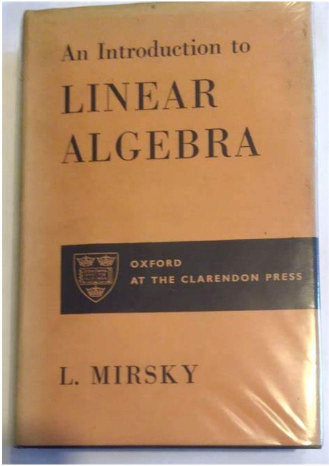
Jacobson(1910–1999)《线性代数》(1953)



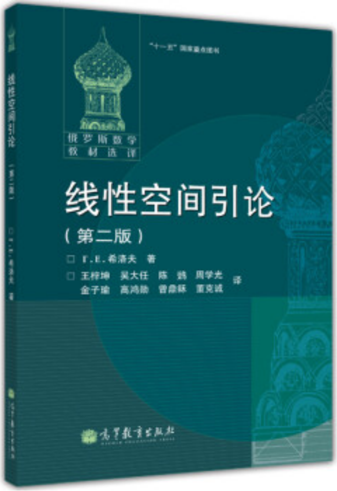
黄缘芳 (1904–1966)

感谢复旦大学华宣积教授提供照片

Mirsky(1918–1983)《线性代数引论》(1955)



希洛夫 (1917–1975)《线性空间引论》(1956)

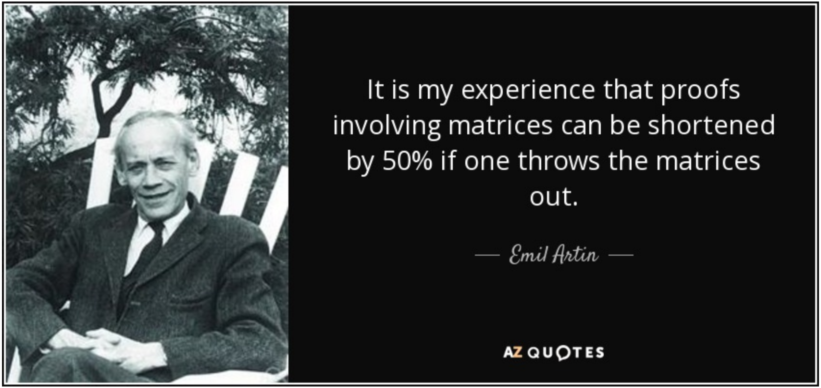


张禾瑞、郝炳新《高等代数》(1957)

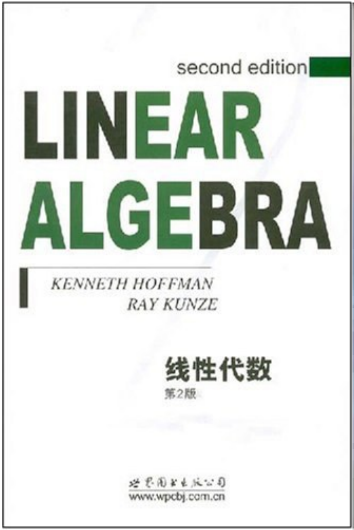
60 年代国内出版的高等代数教材

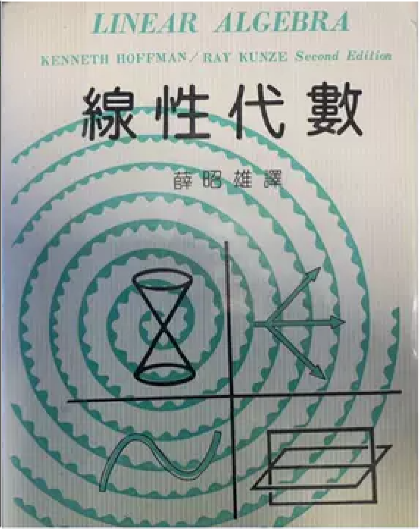
- 王湘浩、谢邦杰《高等代数》(1961)
- 丁石孙《高等代数讲义》(1964)
- 丁石孙《高等代数简明教程》(1966)
- 许以超《代数学引论》(1966)
- 周伯壎《高等代数》(1966)

Artin (1898–1962)

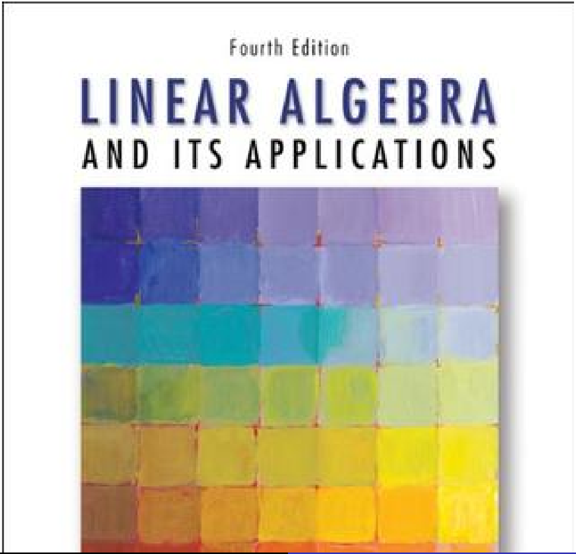


Hoffman–Kunze(1961)

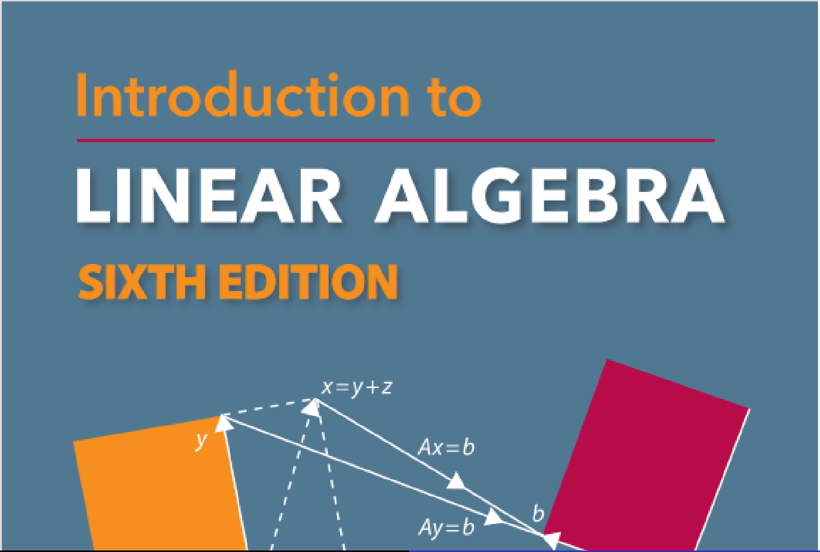




Strang(1976)



Strang(1993)


Introduction to
LINEAR ALGEBRA
SIXTH EDITION
x=y+z
y
Ax=b
b
Ay=b

Strang(1934–)

He made linear algebra fun

A series of numbers describes the career of Professor Gil Strang as he retires from MIT after six highly influential decades on the faculty.

Watch Video

Sandi Miller | Department of Mathematics
May 31, 2023

线性代数基本定理

The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://www.tandfonline.com/loi/uamm20

The Fundamental Theorem of Linear Algebra

Gilbert Strang

To cite this article: Gilbert Strang (1993) The Fundamental Theorem of Linear Algebra, The American Mathematical Monthly, 100:9, 848-855, DOI: 10.1080/00029890.1993.11990500

线性代数教学的核心思想

DOCEAMUS

doceamus...let us teach

The Core Ideas in Our Teaching

Gilbert Strang

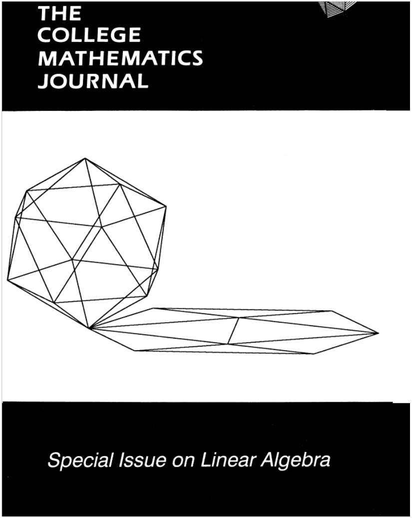
Strang 的三个教学建议

- It is natural to prepare for a class by deciding on a plan. Start with a question that it is important to answer. What is the inverse of a matrix and which matrices have inverses?
- The important point is that in working with simple examples you are giving the students a chance. The key is to build their confidence as active users of mathematics. The teacher has to be saying, in many ways but not in so many words, *you can do it*.
- I would like to emphasize the importance of *the teacher′ s voice*.

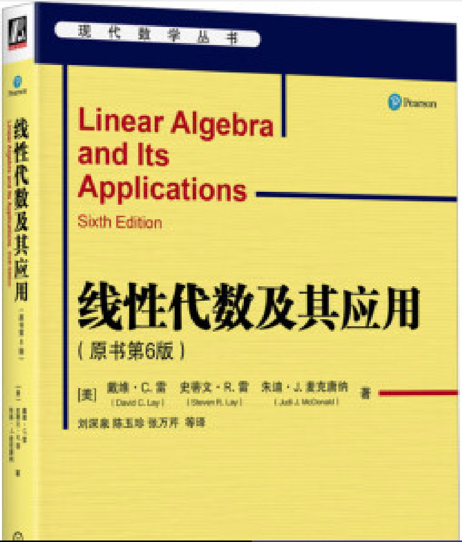
The Linear Algebra Curriculum Study Group (1990, 2018)

- D. Carlson, C. R. Johnson, D. C. Lay, and A. Porter , The linear algebra curriculum study group recommendations for the first course in linear algebra, The College Mathematics Journal 24 (1993), no. 1, 41–46.
- S. Stewart, S. Axler, R. Beezer, E. Boman, M. Catral, G. Harel, J. McDonald, D. Strong, and M. Wawro, The Linear Algebra Curriculum Study Group (LACSG 2.0) Recommendations. Notices of the American Mathematical Society 69 (2022), no. 5, 813–819.

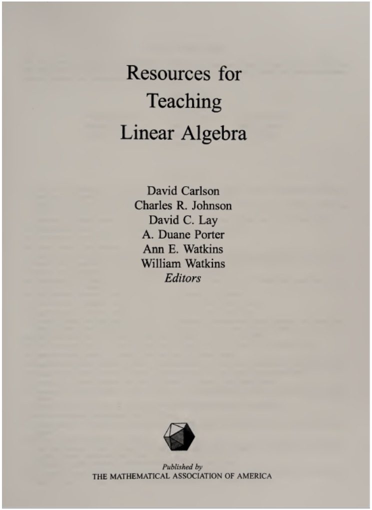
《大学数学杂志》线性代数特刊



Lay 的教材 (1994)


现代数学丛书
Pearson
Linear Algebra and Its Applications
Sixth Edition
线性代数及其应用
(原书第6版)
[美] 戴维·C.雷 史蒂文·R.雷 朱迪·J.麦克唐纳 著
刘深泉 陈玉珍 张万芹 等译

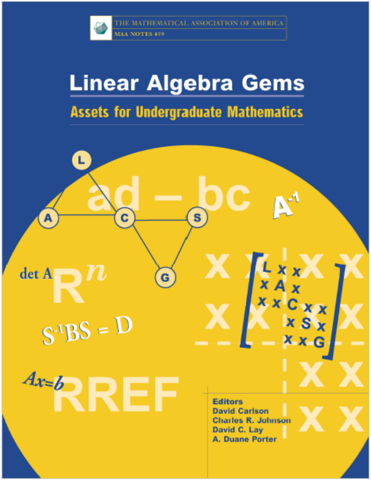
两本教学资料 (1997,2002)



Resources for Teaching Linear Algebra

David Carlson
Charles R. Johnson
David C. Lay
A. Duane Porter
Ann E. Watkins
William Watkins
Editors

Published by
THE MATHEMATICAL ASSOCIATION OF AMERICA



初等线性代数教学大纲

Table 1. Topics appropriate for a first course.

- Systems of linear equations.
- Properties of $\mathbf{R}^n$. Linear independence, span, bases, and dimension.
- Matrix algebra. Column space, row space, null space.
- Linear maps. Matrices of a linear map with respect to bases; the advantages of a change of basis that leads to a simplified matrix and simplified description of a linear map.
- Matrix multiplication and composition of linear maps, with motivation and applications.
- Invertible matrices and invertible linear maps.
- Eigenvalues and eigenvectors.
- Determinant of a matrix as the area/volume scaling factor of the linear map described by the matrix.
- The dot product in $\mathbf{R}^n$. Orthogonality, orthonormal bases, Gram-Schmidt process, least squares.
- Symmetric matrices and orthogonal diagonalization. Singular value decomposition.
- Orthogonal and positive definite matrices

Second Courses in Linear Algebra

A single semester course cannot possibly cover all the important topics in linear algebra. We recommend that students planning to pursue a graduate degree in mathematics or a career in any type of quantitative analysis take at least two courses in linear algebra. Students majoring in computer science, physics, economics, and other subjects using mathematical models may also benefit greatly from two or more courses in linear algebra.

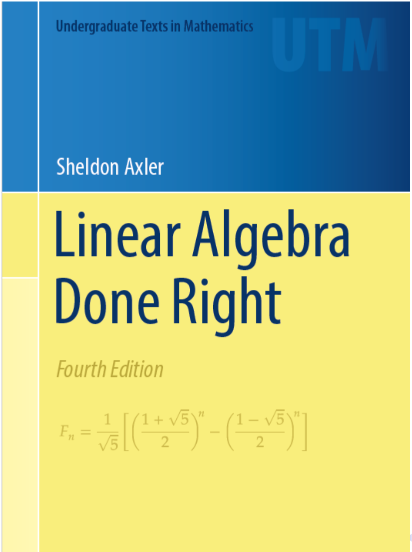
A second course in linear algebra might focus far more on vector spaces, linear maps, and proofs than a typical first course that focuses on $\mathbf{R}^n$, matrices, and computations. The context for the second course should include complex vector spaces (and possibly vector spaces over an arbitrary field) as well as real vector spaces. The following topics (see Table 2) can be included and explored in depth in a second course aimed at math majors and other students who want to learn the powerful tools of linear algebra.

高等线性代数教学大纲

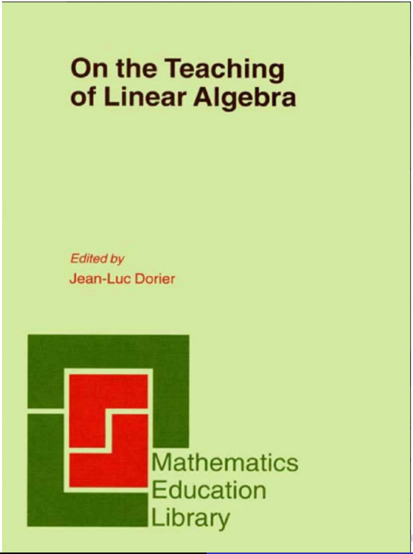
Table 2. Topics appropriate for a second course.

- Abstract vector spaces. Subspaces, linear independence, span, bases, and dimension in the context of abstract vector spaces.
- Linear maps between vector spaces; the null space and range of a linear map. The matrix of a linear map with respect to a basis; rank and nullity of a matrix.
- Invertibility of matrices and linear maps.
- Eigenvalues, eigenvectors, and diagonalization.
- Matrix factorizations.
- Inner product spaces. Orthogonality and orthonormal bases. Gram-Schmidt process for constructing orthonormal bases.
- Orthogonal projections and best approximations. Upper-triangular matrices with respect to an orthonormal basis (Schur's theorem).
- Finite-dimensional spectral theorem. Singular values and singular value decomposition. Low-rank approximation of linear maps. Pseudo-inverses.
- Positive operators. Unitary operators.

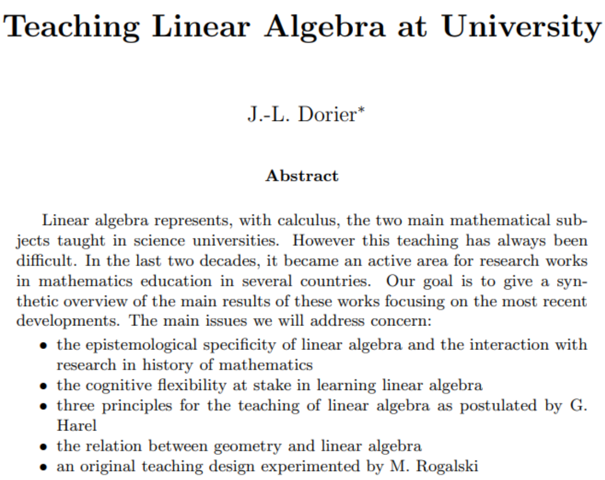
Axler 的高等线性代数教材 (1996)



Dorier1997 年汇编的线性代数教学材料



Dorier 在 2002 年 ICM 上的报告



福建省《高等代数》、《线性代数》课程建设研讨会

逻辑 vs 历史

线性代数本身的发展演变过程非常缓慢艰难，历时 130 多年，包含几个阶段，从我们现在的观点看，这些阶段性的历史发展与概念的逻辑次序刚好相反，即

线性方程组

$\downarrow$

行列式

$\downarrow$

线性型与双线性型

$\downarrow$

矩阵

$\downarrow$

向量空间和线性映射

迪厄多内，《泛函分析史》序言

两篇短文

- Israel Kleiner, History of linear algebra, in *A History of Abstract Algebra*, Boston, Birkhäuser, 2007 .
- L. A. Steen (1973), Highlights in the history of spectral theory, The American Mathematical Monthly, 80:4, 359–381.

向量空间的概念定型

- In 1918, in his book Space, Time, Matter, which dealt with general relativity,Weyl axiomatized finite-dimensional real vector spaces. The definition appears in the first chapter of the book, entitled Foundations of Affine Geometry.
- In his doctoral dissertation of 1920 Banach axiomatized complete normed vector spaces (now called Banach spaces) over the reals. The first thirteen axioms are those of a vector space, in very much a modern spirit.
- In her 1921 paper "Ideal theory in rings" Emmy Noether introduced modules, and viewed vector spaces as special cases.
- We thus see vector spaces arising in three distinct contexts: geometry, analysis, and algebra. In his 1930 classic text Modern Algebra van der Waerden has a chapter entitled Linear Algebra. Here for the first time the term is used in the sense in which we employ it today.

谱定理的历史

- 1637: Descartes, Fermat, $ax^2+2bxy+cy^2$ 可通过旋转化为标准型 $\alpha x^2+\beta y^2$.
- 1748: Euler 讨论了二元、三元二次型的化简
- 1759: Lagrange 讨论了 n 元二次型的化简
- 1765: Euler 引入术语“主轴”
- 1827: Jacobi 研究了种种二次曲面的主轴
- 1829–1830: Cauchy 证明了实二次型的规范型中的系数必为实数
- 1852: Sylverster 证明了那些系数为特征值
- 1858: Cayley 确立了主轴定理的现代形式
- 20 世纪初, Hilbert 引入了术语“谱”，并将主轴定理推广到无穷多个变量

珍宝 1：Birkhoff–Von Neumann 定理与华罗庚引理

定理 (Birkhoff–Von Neumann 定理)

设 V 是域 F 上的向量空间，$B: V \times V \to F$ 是 V 上的双线性型，使得由 $B(x, y) = 0$ 定义的正交关系是对称的，即 $B(x, y) = 0 \iff B(y, x) = 0$，则 B 是对称的或者反对称的。

引理 (华罗庚引理)

设 B 是域 F 上向量空间 V 上的双线性型，使得对任意的 $x, y \in V$ 或者有 $B(x, y) = B(y, x)$ 或者有 $B(x, y) = -B(y, x)$，则 B 是对称的或反对称的。

朱富海教授在其问题引导的代数学问题 7.26 中进一步细化了这个结果.

华罗庚先生跟我讲过，被引用的研究成果可以分为不同的等级．第一个等级是在国外的综述性文章或介绍性文章中被引用，这是一般性引用；第二个等级是在国外学术专著中被引用，并且把具体结果写出来；第三个等级是作为正文被写进教科书中，这是最高级别的引用．

徐利治，《徐利治访谈录》

珍宝 2：中国剩余定理的线性代数版本

引理 (1 在多项式环中的正交幂等分解)

设 $f_1, \ldots, f_n \in K[x]$ 两两互素，令 $f = f_1 \cdots f_n$，
$g_i = \frac{f}{f_i}$，$i = 1, \ldots, n$. 则存在 $u_1, \ldots, u_n \in K[x]$ 使得

- (1 的分解)

$$1 = u_1(x)g_1(x) + \cdots + u_n(x)g_n(x). \tag{1}$$

- (分量正交) 对任意的 $i \neq j$ 有

$$u_i(x)g_i(x)u_j(x)g_j(x) \equiv 0 \pmod{f(x)}. \tag{2}$$

- (分量幂等) 对每个 $i = 1, \ldots, n$，有

$$(u_i(x)g_i(x))^2 \equiv u_i(x)g_i(x) \pmod{f(x)}. \tag{3}$$

向量空间直和分解的算子刻画

引理

设 V 是域 K 上向量空间, $P_1, \ldots, P_n$ 是投影变换, 满足 $P_1 + \cdots + P_n = E$, 且当 $i \neq j$ 时, 有 $P_i P_j = 0$. 则 $V = \bigoplus_{i=1}^n \operatorname{Im}(P_i)$.

中国剩余定理的线性代数版本

定理

设 A 是域 K 上向量空间 V 的线性变换, 对 $f \in K[x]$, 记

$$\ker(f) = \ker(f(A)) = \{v \in V : f(A)v = 0\}.$$

设 $f_1, \ldots, f_n \in K[x]$ 两两互素, 则有向量空间的直和分解

$$\ker(f_1 \cdots f_n) = \ker(f_1) \oplus \cdots \oplus \ker(f_n). \tag{4}$$

而且, 对每个 $i = 1, \ldots, n$, 存在 $e_i \in K[x]$ 使得

$$e_i(A) = P_i, \tag{5}$$

其中 P_i 是 (4) 中的直和到子空间 $\ker(f_i)$ 的投影. 事实上, 可取 $e_i = u_i g_i$ 为前述引理 (1) 中的各个分量, $i = 1, \ldots, n$.

另一件印象比较深刻的事，是上初中时，我对中国古代数学萌发了一些兴趣。记得那时我们在念代数，教科书是《范氏大代数》。那时一直困惑我的一个问题是：为什么我们的数学教科书上没有一个来自中国文明的定理和成就？

程贞一

珍宝 3： Jordan 标准形过渡矩阵的新算法

矩阵的相似标准型的算法和证明是线性代数中最困难的问题。

李尚志，《线性代数（数学专业用）》前言

Jordan 标准形与过渡矩阵的同步求解

设 $P, Q \in \mathrm{GL}_n(F[x])$ 满足

$$P(xI_n - A)Q = \mathrm{diag}(1, \ldots, 1, d_1, \ldots, d_r), \tag{6}$$

其中 $d_1, \ldots, d_r \in F[x]$ 是不变因子, 满足 $d_1 | d_1 | \cdots | d_r$. 记

$$Q = (*, \ldots, *, \xi_1, \ldots, \xi_r), \qquad \xi_i \in F[x]^n. \tag{7}$$

假定给出了各个不变因子 $d_i(x)$ 的因式分解

$$d_i = \prod_{\lambda \in \mathrm{Spec}(A)} (x - \lambda)^{m_i(\lambda)}, \qquad 1 \leqslant i \leqslant r. \tag{8}$$

对每个 $\lambda \in \mathrm{Spec}(A)$ 以及每个使得 $m_i(\lambda) \neq 0$ 的 $i \in \{1, \ldots, r\}$, 令

$$\alpha_{ij}(\lambda) = \langle \xi_i, (x - \lambda)^j \rangle \in F^n, \qquad 0 \leqslant j \leqslant m_i(\lambda) - 1, \tag{9}$$

其中 $\langle \xi_i, (x - \lambda)^j \rangle$ 表示多项式向量 ξ_i 中关于 $x - \lambda$ 的 Taylor 展开式中 $(x - \lambda)^j$ 的系数向量. 则以下断言成立:

(i) 对每个 $\lambda \in \mathrm{Spec}(A)$ 以及每个使得 $m_i(\lambda) \neq 0$ 的 $i \in \{1, \dots, r\}$, 向量

$$\{\alpha_{ij}(\lambda) : 0 \leqslant j \leqslant m_i(\lambda) - 1\} \tag{10}$$

线性无关, 并生成 F^n 的一个 L_A-不变子空间 $W_i(\lambda)$. 限制线性变换 $L_A|_{W_i(\lambda)}$ 在 $W_i(\lambda)$ 的基底 (??) 下的矩阵为 Jordan 块:

$$J_{m_i}(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}. \tag{11}$$

(ii) F^n 是子空间 $W_i(\lambda)$ 的直和. 从而向量

$$\{\alpha_{ij}(\lambda) : \lambda \in \mathrm{Spec}(A), m_i(\lambda) \neq 0, 0 \leqslant j \leqslant m_i(\lambda) - 1\} \tag{12}$$

构成 F^n 的一组基, L_A 在这组基下为 Jordan 标准型.

最早的几何学、最早的方程组、最古老的矩阵等等，翻开历史，中国曾经是一个数学的国度，中国数学在世界上的位置远比今天靠前。祖冲之、刘徽、《九章算术》、《周髀算经》、《四元玉鉴》等一批大家和著作，使中国数学曾经处于世界巅峰。正是由于这些辉煌，中国数学不仅要振兴，更要复兴！

吴文俊

相关工作

- 林开亮，Hua 引理及其应用,《数学传播》，第 34 卷第 3 期 (2010 年)，39–48.
- 林开亮、王兢，中国剩余定理的若干应用,《中央民族大学学报（自然科学版)》，2023 年 8 月，第 32 卷第 3 期，28–33.
- 安金鹏、林开亮、孙亦青，从 $(xI_n - A)$ 的列变换矩阵求 A 的标准型基底，《蛙鸣》第 66 期，2023 年, 5–14. 英文版见安金鹏老师的主页 Construction of Bases for Rational and Jordan Canonical Forms.
- 林开亮，线性代数珍宝十三则,《数学传播》，第 37 卷第 4 期（2013 年)，65–83.

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- Abel–Galois 不可约原理漫谈
- 高等代数与中国古代数学
- 特征矩阵的初等变换与 Jordan 标准基

致谢

- 感谢林亚南教授邀请我参加福建省“高等代数”与“线性代数”课程建设第二十三次研讨会并做报告。
- 感谢福州大学数学与统计学院各位领导和老师的盛情款待!
- 感谢各位专家和老师听我分享。请大家多多指教!